Finding the Impulse Response:
Example 2.3 in Lathi's *Linear Systems and Signals* book (second edition)

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The original differential equation governing a system is of the form

$$(D^2 + 5\,D + 6)[y(t)] = (D + 1)[x(t)]$$

where $Q(D) = D^2 + 5\,D + 6$ and $P(D) = D + 1$.

Characteristic polynomial is $(\lambda + 3)\,(\lambda + 2)$, which leads to the characteristic modes

$$y_0(t) = C_1\,e^{-2\,t} + C_2\,e^{-3\,t}$$

The impulse response from equation (2.23) is

$$h(t) = [\;P(D)\,y_0(t)\;]\,u(t) = [y_0(t) + y_0'(t)]\,u(t)$$

After substituting in the expressions for $y_0(t)$ and $y_0'(t)$,

$$h(t) = [\;(C_1\,e^{-2\,t} + C_2\,e^{-3\,t}) + (-2\,C_1\,e^{-2\,t} - 3\,C_2\,e^{-3\,t})\;]\,u(t) = [-C_1\,e^{-2\,t} - 2\,C_2\,e^{-3\,t}\;]\,u(t)$$

By definition, the impulse response is the response of a system to an impulse. To find the impulse response from the differential equation governing the system, we set $x(t) = \delta(t)$:

$$h''(t) + 5\,h'(t) + 6\,h(t) = \delta(t) + \delta'(t)$$

We can see that impulsive events are occurring at the origin. The impulsive events will lead to a point of discontinuity in $h(t)$ at t=0 and likewise in $h'(t)$ at t=0. The next step is to balance the impulsive events in the impulse response.

For the zero-state response, $h(0^-) = 0$ and $h'(0^-) = 0$. Let $h(0+) = K_1$ and $h'(0+) = K_2$.

Note: Consider a causal signal $f(t)$ that has a point of discontinuity at the origin and the value of $f(0^+)$ is K_1. An example would be $f(t) = K_1\,u(t)$, which has $f(0^-) = 0$ and $f(0^+) = K_1$. Hence, $f'(t) = K_1\,\delta(t)$.

Let's try to find the first and second derivatives of the impulse response at t=0:

$$h'(0) = K_1\,\delta(t)$$
$$h''(0) = K_1\,\delta'(t) + K_2\,\delta(t)$$

Note: The Dirac delta functional $\delta(t)$ is not defined at t=0. Hence, we have to keep the placeholder here.

Let's return to the earlier equation for the impulse response:

$$h''(t) + 5\,h'(t) + 6\,h(t) = \delta(t) + \delta'(t)$$

and analyze the impulse response at t=0:

$$h''(0) + 5\,h'(0) + 6\,h(0) = \delta(t) + \delta'(t)$$

By substituting for $h'(0)$ and $h''(0)$,

$$(\,K_1\,\delta'(t) + K_2\,\delta(t)\,) + 5\,(\,K_1\,\delta(t)\,) + 6\,h(0) = \delta(t) + \delta'(t)$$

By collecting terms, we have

$$(K_2 + 5\,K_1)\,\delta(t) + \;K_1\,\delta'(t) + 6\,h(0) = \delta(t) + \delta'(t)$$

Note: We can define any value we would like to assign to $h(0)$ because $h(t)$ at t=0 is a point of discontinuity.

By balancing the Dirac delta terms and the first-derivative of the Dirac delta terms on the left and right hand sides of the equation, we obtain

$$K_1 = 1$$
$$K_2 + 5\,K_1 = 1 \quad \Rightarrow \quad K_2 = -4$$

Let's now return to solving for C1 and C2:

$$h(0+) = -C_1 - 2\,C_2 = -1 = K_1$$
$$h'(0+) = 2\,C_1 + 6\,C_2 = -4 = K_2$$

Which means that $C_1 = 1$ and $C_2 = -1$.

The solution for the impulse response is

$$h(t) = [-e^{-2\,t} + 2\,e^{-3\,t}\,]\,u(t)$$